Log-Concavity Property of the Error Probability With Application to Local Bounds for Wireless Communications

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Abstract—A clear understanding of the behavior of error probability (EP) as a function of signal-to-noise ratio (SNR) and other system parameters is fundamental for assessing the design of digital wireless communication systems. We propose an analytical framework based on the log-concavity property of the EP which we prove for a wide family of multidimensional modulation formats in the presence of Gaussian disturbances and fading. Based on this property, we construct a class of local bounds for the EP that improve known generic bounds in a given region of the SNR and are invertible, as well as easily tractable for further analysis. This concept is motivated by the fact that communication systems often operate with performance in a certain region of interest (ROI) and, thus, it may be advantageous to have tighter bounds within this region instead of generic bounds valid for all SNRs. We present a possible application of these local bounds, but their relevance is beyond the example made in this paper.

Index Terms—Error statistics, fading channels, local bounds, log-concavity, performance evaluation, probability.

I. INTRODUCTION

T HE performance evaluation for digital wireless communication systems in terms of bit error probability (BEP) and symbol error probability (SEP) requires a careful characterization of disturbances, such as noise and interference, as well as of the wireless channel impairments due to small-scale and large-scale fading (see, e.g., [1]–[3]). This can result in cumbersome expressions for the error probability (EP) which require numerical evaluation.[1]

At a first thought, this fact does not appear a relevant issue from the performance study point of view due to the increasing trend of computational power of computers. On the other hand, these cumbersome solutions do not provide a clear understanding of the performance sensitivity to system parameters, which is of great importance for system design, as well as they

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[1]Hereafter, when EP is indicated without specification of BEP and SEP then the concept is valid for both BEP and SEP.

are often too complicated for further evaluation or inversion, which is as example needed in order to obtain thresholds in adaptive communication systems (see, e.g., [4]–[6]). Moreover, it has to be emphasized that simple parametric approximations and bounds on the performance at lower layers, such as physical layer, can avoid long bit-level simulations in upper level protocols network simulators, provided that they are able to capture the main aspects affecting the performance at lower levels.

Mainly, but not only, for these reasons the derivation of approximations and bounds on the exact EP is still of interest in the communication theory community. An example is given by M-ary quadrature amplitude modulation (M-QAM) that is adopted in several standards for wireless communication systems, due to its bandwidth efficiency, and is largely studied in conjunction with adaptive techniques which change modulation parameters to maximize transmission rate for a given target BEP in wireless channels. In fact, although early work on M-QAM dates back to the early 1960s [7]–[10], the evaluation of BEP for arbitrary M is still of current interest.[2] To briefly summarize some relevant results for additive white Gaussian noise (AWGN) channel, we recall that: parameterized exponential approximations fitting simulative BEP are adopted in [12]–[15]; approximations based on signal-space concepts were given in [16]; an exact method to derive the SEP was proposed in [17]; a recursive algorithm exploiting the relationship among different constellation sizes was developed in [18]; and exact expression of the BEP for general M was derived in [19]. Comparisons among approximations, bounds, and the exact solution in fading channels (with small-scale fading and large-scale fading, i.e., shadowing) are given in [5], where it is shown that, for low and medium values of the signal-to-noise ratio (SNR), approximations depart from exact solutions as the constellation size M increases. Moreover, small differences between exact solution and approximation in AWGN channel can become relevant when the instantaneous BEP is averaged over small-scale fading. Similarly, in systems with multichannel reception, known approximations depart from the exact EP as the diversity order increases [20], [21].

It is well known that bounds carry more information than approximations and also enable system design based on the worst or best case. Quite often bounds are tight to the exact EP only for high SNRs (namely, asymptotic bounds). Here we are interested in deriving simple invertible bounds tight in a given region of interest (ROI) for the EP (e.g., for the BEP of uncoded systems typical ROIs are $[10^{-3}, 10^{-1}]$ or $[10^{-4}, 10^{-2}]$).

[2]For a brief history of M-QAM, see [11].

In this paper, we define the concept of locally valid bounds (in the following called *local bounds*) that are tight upper and lower bounds on the EP valid only within a given region of the EP and not for all SNRs. This concept is motivated by the fact that there is often a ROI for the performance of the system under consideration and it is preferable to have tight bounds in this region instead of bounds valid for all SNRs which are far from the exact solution within the ROI.

The behavior of the EP is important for the definition of local bounds. In fact, the proposed framework is based on its *log-concavity property*. We recall that a function $z(\cdot)$ is log-concave if $\log z(\cdot)$ is concave.[3] In most cases, the EP is reported on log-scale and investigated as a function of the signal-to-total disturbance ratio in decibel (dB). It is commonly recognized that on this scale the function is concave in several cases of interest. Even though it is generally assumed, as the authors often acknowledge, there is not known formal proof of the log-concavity of the EP (examples of related issues are: convexity properties in binary detection problems that were analyzed in [22], and some results for the asymptotic behavior of bounds that were investigated in [23]).

In this paper, we introduce the problem of log-concavity for general multidimensional decision regions and we prove this property for a class of signals with constellation on a multidimensional regular grid in the presence of Gaussian distributed disturbances, such as thermal noise and interference. In fact, there are several wireless systems and situations in which the interference can be modeled as Gaussian distributed (see, e.g., [24]–[29]). After having proved the log-concavity in both AWGN and fading plus AWGN channels for single- and multiple-channel reception schemes, as examples of application, it will be shown how to take advantage of this property in order to simplify the derivation of bounds valid for all SNRs, as well as to define local upper bound (LUB) and local lower bound (LLB) valid in a given ROI. Moreover, the form of the local bounds and the fact that they are easily invertible also enables the derivation of local bounds for other relevant performance figures such as the EP-based outage probability, which is the probability that the EP averaged over small-scale fading exceed a given tolerable target value [30], also exploited for the evaluation of the mean spectral efficiency for adaptive modulation techniques [6]. It is finally emphasized that the log-concavity property for the EP can have many other applications, thus its relevance is beyond the applications illustrated in this paper.

The rest of this paper is organized as follows. In Section II, the log-concavity property of the EP is proved in AWGN and in AWGN plus fading for systems employing single- and multiple-channel reception, and in Section III, it is applied to define a new class of bounds and local bounds, with a discussion on possible applications. Finally, our conclusions are reported in Section IV.

II. LOG-CONCAVITY PROPERTY FOR THE ERROR PROBABILITY

In this section, the log-concavity property for the EP is discussed first for transmission in AWGN channel by highlighting the mathematical structure of the problem in the different applications of digital communications. Since we are interested in obtaining general results, we base our framework on the origin of detection errors in the presence of Gaussian disturbances. Within this framework, we will then prove the log-concavity property for the class of signals with constellation on a multidimensional regular grid (e.g., in the two-dimensional case, this class includes the well-known M-QAM constellation). Finally, we will address the log-concavity property in systems with AWGN plus fading channels.

Consider a set $\mathcal{X}$ of M constellation points on a d-dimensional signal space, i.e., $\mathcal{X} = \{x_i \in \mathbb{R}^d : i = 1, \ldots, M\}$. Let us consider an arbitrary probability distribution on the set $\mathcal{X}$ and let p_i denote the probability of a point $x_i \in \mathcal{X}$ for $1 \leq i \leq M$ (we arbitrarily order points in $\mathcal{X}$ without loss in generality). For each $x_i \in \mathcal{X}$, let us define its neighborhood

$$R_i = \left\{ x \in \mathbb{R}^d : |x - x_i| = \min_j |x - x_j| \right\} \quad (1)$$

as the set of points closest to x_i in $\mathcal{X}$. If we transmit a point x_i in an AWGN channel, then we receive $x_i + \sigma g$ where $\sigma > 0$, and g has a standard Gaussian distribution $\mathcal{N}(0, I)$ on $\mathbb{R}^d$ with mean zero and identity covariance matrix.[4] We classify each point according to a region R_i that it belongs to, which means that we make an error if $x_i + \sigma g \notin R_i$ or

$$g \notin \sigma^{-1}(R_i - x_i) = \left\{ \sigma^{-1}(y - x_i) : y \in R_i \right\}. \quad (2)$$

Through the change of variable[5] $\sigma^{-1} = e^t$, the total probability of making an error results in[6]

$$p(t, \mathcal{X}) = \sum_i p_i \mathbb{P}\left\{ g \notin e^t(R_i - x_i) \right\}$$
$$= 1 - \sum_i p_i \mathbb{P}\left\{ g \in e^t(R_i - x_i) \right\}. \quad (3)$$

If we denote by μ a standard Gaussian measure on $\mathbb{R}$, then the distribution of vector g is a product measure $\mu^{\otimes d}$, and therefore

$$p(t, \mathcal{X}) = \sum_i p_i \mu^{\otimes d}\left(e^t \overline{R_i'} \right) = 1 - \sum_i p_i \mu^{\otimes d}\left(e^t R_i' \right) \quad (4)$$

where we denoted by $R_i' = R_i - x_i$ the region R_i translated by x_i, and by $\overline{R_i'}$ the region $\mathbb{R}^d - R_i'$.

The function $p(t, \mathcal{X})$ is the error probability in the detection of digital signals, either coded or uncoded, as a function of SNR t (in logarithmic scale). Proving the log-concavity property of this function is a challenging task. In fact, the log-concavity of single terms in the linear combination of (4) depends on the specific structure of regions R_i' and $\overline{R_i'}$ and in any case a possible linear combination of log-concave functions is not necessarily log-concave.

Only in few special cases, as example when all the regions R_i' have the same measure and a special symmetry around the axis intersecting points 0 and $-x_i$, these issues may be overcome with the help of the Prekopa–Leindler theorem [31], [32], which states that the function $F(\mathbf{x}) = \int_A f(\mathbf{x}, \mathbf{y}) d\mathbf{y}$, where

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[3]In this paper, notation log stands for natural logarithm.

[4]Note that σ^2 is inversely proportional to the SNR.

[5]Which is strictly related to the transformation of the SNR from the linear to the decibel scale.

[6]Notation $\mathbb{P}\{\mathcal{A}\}$ stands for probability of event $\mathcal{A}$.
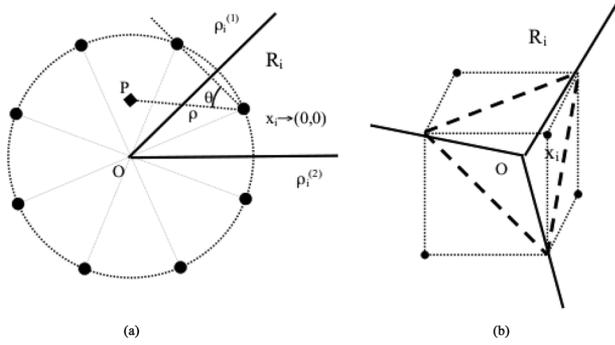
Fig. 1. Decision regions R_i for (a) Example 1 and (b) Example 2 of Section II, respectively.

$\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m$, is log-concave in $\mathbb{R}^n$ if $f(\mathbf{x}, \mathbf{y})$ is log-concave in $\mathbb{R}^{n+m}$ and A is a convex subset of $\mathbb{R}^m$.

Two examples, one for uncoded system and the other for coded system, will illustrate these simple cases below. In all the other cases, an inspection of log-concavity property should be based on the specific properties of the signal set $\mathcal{X}$. In Section II-A, we will provide the proof of log-concavity property for the specific case of signals with constellation on a multidimensional regular grid, which covers all the relevant applications based on M-QAM signaling.

Example 1 (M-PSK): Let us consider a two-dimensional signal set $\mathcal{X}$ where M points are regularly placed on a circle. The angular separation between closest points is $2\pi/M$ [see Fig. 1(a)]. This is the signal set used by M-phase shift keying (PSK) signaling. The regions R_i' are circular sectors, have the same form and the same measure, and are convex. The same holds for regions $\overline{R_i'}$, which are concave instead. If we split each region $\overline{R_i'}$ in two parts, $\rho_i^{(1)}$ and $\rho_i^{(2)}$, using the line including the points $(0,0)$ and $-x_i$ (i.e., point 0 in the figure), all these subregions have the same Gaussian measure and are convex. Since $p(t, \mathcal{X}) = 2\mu^{\otimes 2}(e^t \rho_i^{(1)})$, the log-concavity of a single term has to be checked. The two-dimensional Gaussian measure can be evaluated by using polar coordinates[7] in $\mathbb{R}^2$ as

$$\mu^{\otimes 2}\left(e^t \rho_i^{(1)}\right) = \frac{1}{2\pi} \int_{-\pi/2+\pi/M}^{\pi/2} e^{-S^2(\theta)e^t} d\theta \qquad (5)$$

where $S(\theta) = \sin(\pi/M)/\cos(\theta)$ describes the boundary of region $\rho_i^{(1)}$. Since the function $e^{-s^2(\theta)e^t}$ is log-concave for $(t, \theta) \in \mathbb{R} \times [-\pi/2 + \pi/M, \pi/2]$, the Prekopa–Leindler theorem[8] assures that $p(t, \mathcal{X})$ is log-concave with respect to t.

Example 2 (Parity Check Linear Block Codes and BPSK): Let us consider the $(k+1)$-dimensional signal set $\mathcal{X}$ representing signals obtained by combining a simple parity-check $(k+1, k)$ binary block code and binary antipodal modulation. All the $M = 2^k$ points are placed on (half of) the vertices of a $(k+1)$-dimensional cube and are equidistant from the origin. Each point has $n_c = k(k+1)/2$ closest points or neighbors and

each region R_i is bounded by n_c faces in the $(k+1)$-dimensional space [see Fig. 1(b)]. All regions R_i' have the same form and the same measure, and are convex. The same holds for regions $\overline{R_i'}$, which are concave instead. Let us now simplify the example to $k = 2$ for better understanding. We have four equidistant points placed on four vertices of a cube. Regions R_i' and $\overline{R_i'}$ are bounded by $nc = 3$ planes intersecting in $-x_i$. If we split each region $\overline{R_i'}$ into three parts, $\rho_i^{(1)}$, $\rho_i^{(2)}$, and $\rho_i^{(3)}$, using three half-planes generated by the line connecting the points $(0,0,0)$ and $-x_i$, all these subregions have the same Gaussian measure and are convex. Since $p(t, \mathcal{X}) = 3\mu^{\otimes 3}\left(e^t \rho_i^{(1)}\right)$, the log-concavity of a single term has to be checked. By using cylindrical coordinates[9] in $\mathbb{R}^3$, the three-dimensional Gaussian measure can be evaluated as

$$\mu^{\otimes 3}\left(e^t \rho_i^{(1)}\right) = \frac{1}{\pi} \int_{-\pi/2+\beta}^{\pi/2} Q\left(\sqrt{(1+S^2(\theta))e^t 4/3}\right) d\theta \qquad (6)$$

where $zS(\theta) = z/(\sqrt{3}\cos(\theta))$ with $z \geq \sqrt{2/3}$ describes the boundary of region $\rho_i^{(1)}$, $\beta = \arcsin(\sqrt{2/3})$, and $Q(.)$ is the Gaussian Q-function. Since the function $Q(\sqrt{(1+S^2(\theta))e^t 4/3})$ is log-concave[10] for $(t, \theta) \in \mathbb{R} \times [-\pi/2 + \arcsin(\sqrt{2/3}), \pi/2]$, the Prekopa–Leindler theorem assures that $p(t, \mathcal{X})$ is log-concave with respect to t.

A. Log-Concavity Property: Signals With Constellation on a Multidimensional Grid in AWGN

Given $a > 0$, consider a set $\mathcal{X}$ of $M = n^d$ points on $\mathbb{R}^d$

$$\mathcal{X} = \{(k_1 a, \dots, k_d a) : k_l = 1, \dots, n \text{ for all } l \leq d\} \qquad (7)$$

that form a regular finite grid on $\mathbb{R}^d$ with each coordinate taking n possible values $a, 2a, \dots, na$.[11] Since $\mathcal{X}$ is a regular grid, all sets R_i' take a particularly simple form, namely, each such set is equal to one of the $d+1$ sets given by

$$[-a/2, \infty)^{d-k} \times [-a/2, a/2]^k, \qquad \text{for } k = 0, \dots, d \qquad (8)$$

up to a permutation of coordinates. The product measure $\mu^{\otimes d}$ is invariant under permutation of coordinates, thus we can identify each set R_i' with one of the sets in (8). If P_k is the sum of probabilities p_i of points contained in the regions of type k, then we obtain

$$\begin{aligned} p(t, \mathcal{X}) &= 1 - \sum_{k=0}^{d} P_k \\ &\quad \times \mu^{\otimes d}\left([-(a/2)e^t, \infty)^{d-k} \times [-(a/2)e^t, (a/2)e^t]^k\right) \\ &= 1 - \sum_{k=0}^{d} P_k \mu\left([-(a/2)e^t, \infty)\right)^{d-k} \\ &\quad \times \mu\left([-(a/2)e^t, (a/2)e^t]\right)^k. \end{aligned} \qquad (9)$$

We note that making a change of variables $t \to (t - \log(a/2))$ it suffices to consider the case of $a/2 = 1$. Let us now define

$$g := g(e^t) \triangleq \mu\left([-e^t, \infty)\right) \qquad (10a)$$

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[7]The origin is the point x_i and θ is the angle with respect to the line orthogonal to $\rho_i^{(1)}$ boundary.

[8]Here, the domain is restricted to $\mathbb{R} \times [-\pi/2, \pi/2]$.

[9]Here, the origin is the point x_i, z is the coordinate along the line orthogonal to $\rho_i^{(1)}$ boundary, and θ is the angle on the plane orthogonal to z-axis.

[10]Note that $Q(\sqrt{x})$ is log-concave, whereas $(1 + S^2(\theta))e^t$ is convex.

[11]Without loss of generality $\mathcal{X}$ can be translated.

$$h := h(e^t) \triangleq \mu\left(\left[-e^t, e^t\right]\right). \tag{10b}$$

This leads to the following representation:

$$p(t, \mathcal{X}) = 1 - \sum_{k=0}^{d} P_k g^{d-k} h^k. \tag{11}$$

Since $g = (h+1)/2$, we obtain

$$p(t, \mathcal{X}) = 1 - H\left(h\left(e^t\right)\right) = \eta(n, d, t) \tag{12}$$

where

$$H(h) \triangleq \sum_{k=0}^{d} P_k \left(\frac{h+1}{2}\right)^{d-k} h^k = \sum_{k=0}^{d} P_k H_k(h). \tag{13}$$

Remark: Notice that all derivatives of H with respect to h are nonnegative and $H(1) = 1$.

We now prove the log-concavity of $\eta(n, d, t)$ with respect to t starting with two lemmas. The main theorem with proof will follow.

Lemma 1: For any $h \in [0, 1]$ and with $H(h)$ given by (13), the following inequality holds:

$$(1 - h) H'(h)^2 - (1 - H(h))\left(H'(h) - (1-h) H''(h)\right) \geq 0. \tag{14}$$

Lemma 2: If $d > 1$, for any $h \in [0, 1]$ and with $H_k(h)$ given by (13), the following inequality holds:

$$(1 - h) H_k'(h) H_m'(h)$$
$$- [1 - H_k(h)]\left[H_m'(h) - (1-h) H_m''(h)\right] \geq 0. \tag{15}$$

We will prove these lemmas in the Appendix and now show how Lemma 1 implies main Theorem 1.

Theorem 1: For any $n, d \geq 1$, the function $t \to \eta(n, d, t)$ is log-concave.

Proof (of Theorem 1): Let $G(e^t) = H(h(e^t))$. A simple calculation gives

$$\frac{d^2}{dt^2} \log\left(1 - G\left(e^t\right)\right)$$
$$= -e^t \frac{(1 - G(e^t)) G'(e^t) + e^t G'(e^t)^2 + e^t (1 - G(e^t)) G''(e^t)}{(1 - G(e^t))^2}. \tag{16}$$

The right-hand side is negative if and only if

$$(1 - G(c)) G'(c) + c G'(c)^2 + c(1 - G(c)) G''(c) \geq 0 \tag{17}$$

where $c = e^t > 0$. Since

$$G'(c) = H'(h) h'(c)$$
$$G''(c) = H''(h) h'(c)^2 - H'(h) c h'(c)$$

and by definition of $h(c)$ giving

$$h'(c) = \sqrt{\frac{2}{\pi}} e^{-\frac{c^2}{2}}$$

$$h''(c) = -c\, h'(c)$$

we can rewrite (17) as

$$(1 - H)\left(H'h'\left(1 - c^2\right) + c(h')^2 H''\right) + c(h')^2 (H')^2 \geq 0$$

or, equivalently

$$c\, h' \geq \left(c^2 - 1\right) \frac{(1 - H) H'}{H''(1 - H) + (H')^2}. \tag{18}$$

It is immediate to see that if $c \leq 1$, then the inequality holds. For $c > 1$, the proof follows from

$$\frac{c\, h'}{c^2 - 1} \geq 1 - h \geq \frac{(1 - H) H'}{H''(1 - H) + (H')^2}. \tag{19}$$

Here, the left-hand side inequality can be derived from

$$\frac{c}{c^2 - 1} e^{-\frac{c^2}{2}} \geq \int_c^\infty e^{-\frac{x^2}{2}} dx \tag{20}$$

which is verified[12] for $c > 1$, and the right-hand side inequality follows from Lemma 1. $\qquad\square$

Remark: The instantaneous BEP expression for coherent single reception M-QAM systems with arbitrary M as a function of the instantaneous symbol SNR γ is given by

$$P_b(e|\gamma) = \frac{2}{\sqrt{M} \log_2 M} \sum_{k=1}^{\log_2 \sqrt{M}} \sum_{i=0}^{(1-2^{-k})\sqrt{M}-1} (-1)^{\left\lfloor \frac{i \cdot 2^{k-1}}{\sqrt{M}} \right\rfloor}$$
$$\times \left(2^{k-1} - \left\lfloor \frac{i \cdot 2^{k-1}}{\sqrt{M}} + \frac{1}{2} \right\rfloor\right)$$
$$\times \operatorname{erfc}\left[(2i+1)\sqrt{\frac{3\gamma}{2(M-1)}}\right] \tag{21}$$

where $\lfloor x \rfloor$ denotes the integer part of x [19]. One might consider to try to prove log-concavity directly using this explicit expression. However, this seems to be a difficult task since sum of log-concave functions is not log-concave in general and the BEP is a linear combination of positive and negative terms containing the complementary error function,[13] $\operatorname{erfc}(\cdot)$, making the analysis of (21) not at all straightforward.

B. Log-Concavity Property: Signals in AWGN Plus Fading

In the above proof of log-concavity for the function $t \to p(t, \mathcal{X})$ the size of the grid $a > 0$ was fixed. When we transmit a symbol related to the constellation point x_i with fixed a in AWGN plus fading channel, the receiver observes $F x_i + \sigma g$ where F is a random variable (RV) representing the channel gain due to fading. This is equivalent to the observation of $x_i + \sigma g$ when the constellation has a random scaling parameter a

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[12]Both sides of (20) tend to 0 as $c \to \infty$, therefore, it is enough and simple to show that for any $c > 1$

$$\frac{d}{dc}\left(\frac{c}{c^2 - 1} e^{-\frac{c^2}{2}}\right) \leq \frac{d}{dc}\left(\int_c^\infty e^{-\frac{x^2}{2}} dx\right).$$

[13]The complementary error function is in well-known relationship with Gaussian Q-function, i.e., $Q(x) = (1/2) \operatorname{erfc}\left[x/\sqrt{2}\right]$.

with the same statistics of the fading gain F.[14] Thus, we show now that when the constellation $\mathcal{X}$ is scaled with a real random parameter a, the average of $p(t, a\mathcal{X})$ over a is still log-concave in t.

Let us now make the dependence of p on a explicit, through the change of variable $b = 2 \log a$, thus $a/\sigma = e^{t+b/2}$. We obtain the function $p(t + b/2, \mathcal{X})$, which is log-concave as a function of both variables (t, b) if $p(t, \mathcal{X})$ is log-concave with respect to t. Hence, what follows is valid for all signal sets $\mathcal{X}$ with log-concave instantaneous EP function of t. To obtain the EP averaged over fading we have to evaluate the expected value of $p(t + b/2, \mathcal{X})$ with respect to the RV b. $\square$

Theorem 2: If b has a log-concave probability density function (pdf), then the average of p over b, that is $\overline{p}(t, \mathcal{X}) = \mathbb{E}_b\{p(t + b/2, \mathcal{X})\}$, is also log-concave.

Proof (of Theorem 2): Suppose that b has a distribution with log-concave density, that is density of the form $e^{-V(b)}$ for some convex function $V(b)$. Then[15]

$$\overline{p}(t) = \int p(t + b/2)e^{-V(b)}db$$

is the average of p over b. Since $p(t+b/2)e^{-V(b)}$ is log-concave in both variables (t, b), Prekopa–Leindler inequality [31], [32] implies that $\overline{p}(t)$ is log-concave. $\square$

Theorem 2 shows that if the distribution of b has log-concave density, then the average over a is log-concave. This applies to several cases of interest (e.g., single- and multiple-channel reception in Rayleigh, Nakagami-m, and log-normal fading) thus leading to log-concave average EP. This can be verified by considering that if the pdf $f_{a^2}(\xi)$ of a^2 is given, then the pdf of $b = \log a^2$ results

$$f_b(z) = e^z f_{a^2}(e^z). \tag{22}$$

For Nakagami-m fading (having $m \geq 1/2$) the pdf of a^2 is given by[16]

$$f_{a^2}(\xi) = \frac{m^m}{\Gamma(m)} \xi^{m-1} e^{-m\xi} \tag{23}$$

from which by (22), we obtain

$$f_b(z) = \frac{m^m}{\Gamma(m)} e^{m(z - e^z)} \tag{24}$$

that is log-concave in z since $m(z - e^z)$ is concave and $m^m/\Gamma(m) > 0$. For log-normal fading (i.e., a^2 in decibels is a zero-mean Gaussian RV with variance σ_{dB}^2) the pdf of a^2 is given by ($\nu = 10/\log 10$)

$$f_{a^2}(\xi) = \frac{\nu}{\sqrt{2\pi}\sigma_{\mathrm{dB}}\xi} e^{-\frac{(10 \log_{10}\xi)^2}{2\sigma_{\mathrm{dB}}^2}} \tag{25}$$

from which by (22), we obtain

$$f_b(z) = \frac{\nu}{\sqrt{2\pi}\sigma_{\mathrm{dB}}} e^{-\frac{\nu^2}{2\sigma_{\mathrm{dB}}^2}z^2} \tag{26}$$

that is log-concave in z since σ_{dB} is positive. For maximal ratio combining of N-branches independent identically distributed (i.i.d.) Rayleigh fading, the pdf of a^2 at the combiner output is given by

$$f_{a^2}(\xi) = \frac{1}{(N-1)!} \xi^{N-1} e^{-\xi} \tag{27}$$

from which by (22), we obtain

$$f_b(z) = \frac{1}{(N-1)!} e^{Nz - e^z} \tag{28}$$

that is log-concave in z since N is positive and $Nz - e^z$ is concave.

It is also important to remark that the log-concavity property for the EP proved above can have several applications, thus its relevance is beyond what is illustrated in Section III, where an application example for bounds and local bounds is provided.

III. BOUNDS AND LOCAL BOUNDS ON LOG-CONCAVE ERROR PROBABILITY

In this section, it is shown how to take advantage of the log-concavity property for the derivation of bounds and local bounds, which are analytically simple and invertible for further analysis. An example of application will be briefly discussed, addressing local bounds of relevant performance metrics for adaptive M-QAM systems. However, the application of the log-concavity is not limited to these cases (e.g., bounds for multidimensional modulations[17] as well as for M-PSK can also be derived). The main idea to be exploited is that, due to the log-concave behavior proved in Section II, the EP plotted in logarithmic scale versus the signal-to-total-disturbance ratio γ in decibels is a concave function (see, e.g., Fig. 2). After having identified the ROI, where the system typically operates, we aim to easily obtain tighter analytically tractable and invertible upper and lower bounds valid in the ROI. The ROI is defined as the range $[P_{\mathrm{em}}, P_{\mathrm{eM}}]$ of the EP which is of interest in the specific application.

With the purpose to make a concrete example, in the following, we consider the case of AWGN plus fading channel in which the performance is defined in terms of mean EP, the EP hereafter, averaged over small-scale fading as a function of the mean γ, that is $\overline{\gamma}$. Since the EP is monotonically decreasing in $\overline{\gamma}$, the ROI corresponds to the SNR range $[\overline{\gamma}_{\mathrm{M}}^{\star}, \overline{\gamma}_{\mathrm{m}}^{\star}]$, with $P_e(\overline{\gamma}_{\mathrm{m}}^{\star}) = P_{\mathrm{em}}$ and $P_e(\overline{\gamma}_{\mathrm{M}}^{\star}) = P_{\mathrm{eM}}$.

Let us first consider bounds valid for all SNRs, that is for a ROI corresponding to SNR in the range $[0, \infty)$. This ROI includes asymptotic behavior of EP. As an example, it is worthwhile to recall that in several cases, such as in single- and multiple-channel reception fading channel with Rayleigh or Nakagami-m pdf, the system achieves a diversity $\mathcal{D}$ if the asymptotic error probability is log-linear. This means that $P_e(\overline{\gamma}) \approx$

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[14]For example, this represents the case of flat fading channel and coherent reception.

[15]We omit here the dependence on set $\mathcal{X}$.

[16]It is well known that Rayleigh fading is included in Nakagami-m when $m = 1$.

[17]See, e.g., [33] and [34]. The benefit provided by multidimensional constellations has been widely known in the design of coded modulation [35], [36].
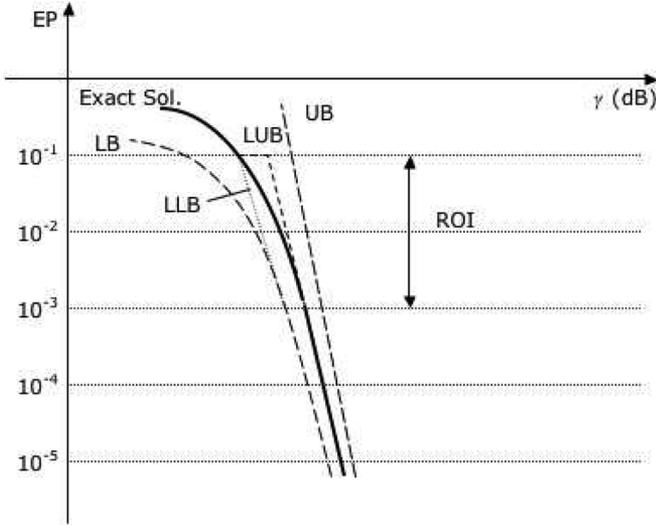
Fig. 2. General behavior for the EP in log-scale versus γ in decibels: concave exact solution, upper and lower bounds (UB, LB) as well as local upper and lower bounds (LUB, LLB) in the ROI of interest are reported.

$v(\overline{\gamma}) \triangleq K/\overline{\gamma}^{\mathcal{D}}$ where K is a constant depending on the asymptotic behavior. In other words, a system with diversity $\mathcal{D}$ is described by a curve of error probability with a slope approaching $10/\mathcal{D}$ (decibel per decade) for large $\overline{\gamma}$.

Thus, we focus our attention on systems with log-linear asymptotical mean EP. For these systems, the log-concavity of the mean EP immediately implies that its asymptotic behavior provides an upper bound in the ROI $[0, \infty)$. Let us consider the usual EP $P_e(\overline{\gamma})$ and the asymptote $v(\overline{\gamma})$, both in logarithmic scale as a function of $\overline{\gamma}$ (in decibels). Note that on this scale the EP is concave whereas $v(\overline{\gamma})$ is linear. It is clear that $P_e(0) \leq v(0)$, since the EP is less than or equal to $1/2$. Furthermore, since $P_e(\overline{\gamma})$ and $v(\overline{\gamma})$ are both decreasing, and the two curves approach at $\overline{\gamma} \to \infty$, then $P_e(\overline{\gamma}) \leq v(\overline{\gamma})$. Therefore, an upper bound to $P_e(\overline{\gamma})$ can be easily defined as

$$P_{e\text{UB}}(\overline{\gamma}) \triangleq \min\left\{\frac{1}{2}, v(\overline{\gamma})\right\}. \tag{29}$$

The UB on the inverse EP, that is on the value of $\overline{\gamma}$ required to reach a target EP $P_e^\star < 1/2$, is thus given by

$$\overline{\gamma}_{\text{UB}}^\star = \left(\frac{K}{P_e^\star}\right)^{1/\mathcal{D}}. \tag{30}$$

To define local bounds, let us now consider a generic ROI $[P_{e\text{m}}, P_{e\text{M}}]$ with $0 < P_{e\text{m}} < P_{e\text{M}} < 1/2$. By shifting the asymptotic UB to touch the exact solution at extremes of the ROI, we define local bounds tighter than previously known bounds, easily invertible, and thus enabling further analysis. At the lower end of the ROI, that is for a target EP $P_e^\star = P_{e\text{m}}$, we can define $\Delta\overline{\gamma}_m$ (in decibels) as the difference between the required $\overline{\gamma}$ for the asymptotic upper bound and the exact solution

$$\Delta\overline{\gamma}_m(\text{dB}) = \overline{\gamma}_{\text{UB}}^\star(\text{dB}) - \overline{\gamma}_{\text{m}}^\star(\text{dB}). \tag{31}$$

Then, in linear scale

$$\Delta\overline{\gamma}_m = \frac{\overline{\gamma}_{\text{UB}}^\star(P_{e\text{m}})}{\overline{\gamma}_{\text{m}}^\star}. \tag{32}$$

We now define the LUB in the ROI as

$$P_{e\text{LUB}}(\overline{\gamma}) \triangleq \min\{P_{e\text{M}}, P_{e\text{UB}}(\overline{\gamma} \cdot \Delta\overline{\gamma}_m)\} \tag{33}$$

which is an invertible upper bound within the ROI. In fact, for a target EP $P_e^\star$ in the ROI, the LUB on the required $\overline{\gamma}$ becomes

$$\overline{\gamma}_{\text{LUB}}^\star = \frac{\overline{\gamma}_{\text{UB}}^\star(P_e^\star)}{\Delta\overline{\gamma}_m} = \overline{\gamma}_{\text{m}}^\star \left(\frac{P_{e\text{m}}}{P_e^\star}\right)^{1/\mathcal{D}}. \tag{34}$$

Thus, to define the LUB one needs to know the exact required SNR at one point, namely, at the lower end of the ROI.

Similarly, one can define the invertible LLB, which is a lower bound within the ROI, by shifting the UB of $\Delta\overline{\gamma}_M$ referred to $P_{e\text{M}}$. This needs only the knowledge of the required SNR for the EP at the upper end of the ROI $\overline{\gamma}_{\text{M}}^\star$. The LLB is given by

$$P_{e\text{LLB}}(\overline{\gamma}) \triangleq \max\{P_{e\text{m}}, P_{e\text{UB}}(\overline{\gamma} \cdot \Delta\overline{\gamma}_M)\}. \tag{35}$$

The LLB on the required $\overline{\gamma}$ results in

$$\overline{\gamma}_{\text{LLB}}^\star = \frac{\overline{\gamma}_{\text{UB}}^\star(P_e^\star)}{\Delta\overline{\gamma}_M} = \overline{\gamma}_{\text{M}}^\star \left(\frac{P_{e\text{M}}}{P_e^\star}\right)^{1/\mathcal{D}}. \tag{36}$$

A qualitative example of bounds and local bounds within the ROI is reported in Fig. 2. At this point, it is important to emphasize that, while the LUB is, within the ROI, a tighter bound than the asymptotic UB and still invertible, the LLB obtained by translation of the UB can be worse compared to known LB, but on the other hand, the LLB is easily invertible enabling further analysis.[18]

Remark: The log-concavity property opens the way for the definition of other classes of bounds, such as based on tangent in the extremes of the ROI or based on saddle point (steepest descent method). Local bounds here proposed have the advantage of being simple and analytically tractable for further analysis.

We now discuss a possible application of local bounds on direct and inverse BEP to the evaluation of relevant performance metrics for adaptive M-QAM systems. Let us consider as an example M-QAM with coherent detection and N-branches MRC, whose exact BEP averaged over i.i.d. Rayleigh fading is given in [6] and its asymptotic upper bound is in the form

$$P_{\text{bUB}}(\overline{\gamma}) \triangleq \min\left\{1/2, T_N(M)/\overline{\gamma}^N\right\}$$

where $T_N(M)$ depends only on the constellation size and the diversity order. From $P_{\text{bUB}}(\overline{\gamma})$, one can obtain the upper bound on the inverse BEP, which is the bound on the SNR required to achieve a target BEP equal to $P_{\text{b}}^\star < 1/2$, and from this invertible LUB and LLB through (34) and (36), respectively. This enables the derivation of LLB and LUB on the error outage (EO), outage probability based on the EP [30], [39], which is an appropriate QoS measure for digital mobile radio when small-scale fading is superimposed on shadowing (typically modeled with

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[18]Asymptotic expressions for the EP in the form of $v(\overline{\gamma})$ can be found using "systematic" approaches when exact EP expressions are not available or asymptotic expressions cannot be easily deduced from well-know (but often complicated) expressions (see, e.g., [37] and [38]).

log-normal distribution [1], [40]).[19] In systems with slow adaptive modulation,[20] for a given target BEP $P_{\rm b}^{\star}$, the spectral efficiency (SE) is a discrete RV with distribution that depends on the SNR thresholds and on how they are computed (i.e., on the BEP expression of the given system configuration). Let M_j and $\overline{\gamma}_{{\rm dB},j}^{\star}$ be the jth element from the set of possible constellation sizes and corresponding SNR threshold (in decibels), respectively, to achieve a target BEP. Then, the mean SE results in

$$\eta = \sum_{j=0}^{J-1} \tilde{M}_j \left[F_{\overline{\gamma}_{\rm dB}}(\overline{\gamma}_{{\rm dB},j+1}^{\star}) - F_{\overline{\gamma}_{\rm dB}}(\overline{\gamma}_{{\rm dB},j}^{\star}) \right]$$
$$+ \tilde{M}_J \left[1 - F_{\overline{\gamma}_{\rm dB}}(\overline{\gamma}_{{\rm dB},J}^{\star}) \right] \quad (37)$$

where $\tilde{M}_k = \log_2 M_k$ and $F_{\overline{\gamma}_{\rm dB}}(\cdot)$ is the cumulative distribution function (CDF) of $\overline{\gamma}_{\rm dB} = 10 \log_{10} \overline{\gamma}$. By substituting in (37) the required SNRs, $\overline{\gamma}_{{\rm dB},j}^{\star}$ with LUB, $\overline{\gamma}_{{\rm LUB\ dB},j}^{\star}$, we obtain an LLB on the mean SE allowing a conservative design of the communication system with different constellation-sizes and diversity orders.

IV. CONCLUSION

In this work, we proved an important property of the error probability as a function of SNR in decibels for AWGN channel as well as AWGN plus fading channels with single- and multiple-channel reception. In particular, we proved that the error probability is log-concave for a wide class of multidimensional modulation formats which include M-QAM for two dimensions. This property can have several applications. As an example, we exploited log-concavity to derive upper and lower bounds and to define local bounds that are tight in a given region of interest for the error probability. We also discussed an application of local bounds highlighting the possibility of easy computation for the inverse of EP formulas without loosing significant accuracy in the evaluation of figures of merit interesting in wireless communications. However, we believe that the relevance of log-concavity property goes beyond the example provided in this paper and may be exploited for other different purposes.

APPENDIX

Proof of Lemma 1: Inequality (14) holds for $h = 1$, hence, from now on, we will assume that $0 \le h < 1$. First, let us prove this inequality in the one-dimensional case $d = 1$. In this case

$$H(h) = P_0 \frac{h+1}{2} + P_1 h$$
$$H'(h) = \frac{1}{2} P_0 + P_1$$
$$H''(h) = 0.$$

Therefore, we need to prove that $(1-h) H' \ge 1 - H$ or

$$(1-h) \left(\frac{1}{2} P_0 + P_1 \right) \ge P_0 \frac{1-h}{2} + P_1 (1-h)$$

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[19]The EO becomes the bit EO (BEO) or the symbol EO (SEO) when respectively related to the BEP or the SEP.

[20]What follows is also valid for fast adaptive modulation for which instantaneous EP and SNR are considered instead of those averaged over small-scale fading [4], [6].

which results in the exact equality. Let us now consider the case $d > 1$. We write the factor $1 - H$ in the second term in (14) as

$$1 - H(h) = \sum_k P_k \left(1 - \left(\frac{h+1}{2} \right)^{d-k} h^k \right)$$
$$= \sum_k P_k (1 - H_k(h))$$

and let us think of the left-hand side of (14) as a homogeneous quadratic form in $(P_k)_{0 \le k \le d}$ of type

$$\sum_{0 \le k,m \le d} P_k P_m H_{k,m}(h) \ge 0$$

where $H_{k,m}(h)$ is given by

$$H_{k,m}(h) = (1-h) H_k'(h) H_m'(h)$$
$$- [1 - H_k(h)] [H_m'(h) - (1-h) H_m''(h)]. \quad (38)$$

Lemma 1 then follows from Lemma 2. $\qquad \square$

Proof of Lemma 2: Let us start by recalling the following well-known identities involving binomial coefficients:

$$\sum_{\alpha=0}^{N} \binom{N}{\alpha} = 2^N \quad (39)$$

$$\sum_{\alpha=0}^{N} \alpha \binom{N}{\alpha} = 2^{N-1} N \quad (40)$$

$$\sum_{\alpha=0}^{N} \alpha^2 \binom{N}{\alpha} = 2^{N-2} N (N+1). \quad (41)$$

As for notation, if $\mathcal{L}$ is a linear combination of $(P_k)_{0 \le k \le d}$, we denote with $\{\mathcal{L}\}_k$ the coefficient of P_k in $\mathcal{L}$. By definition of $H_{k,m}$ in (39) to finish the proof, it is enough to show that for any $0 \le k, m \le n$

$$(1-h) \{H'\}_k \{H'\}_m \ge \{1-H\}_m \{H' - (1-h)H''\}_k. \quad (42)$$

Since

$$\{H\}_k = g^{d-k} h^k = \left(\frac{h+1}{2} \right)^{d-k} h^k \quad (43)$$

we have

$$\{H'\}_k = \left(\left(\frac{h+1}{2} \right)^{d-k} h^k \right)' = \{H\}_k \frac{hd+k}{h(h+1)} \quad (44)$$

and

$$\{H''\}_k = \{H'\}_k \frac{hd+k}{h(h+1)} + \{H\}_k \left(\frac{hd+k}{h(h+1)} \right)'$$
$$= \{H\}_k \frac{(k^2-k) + 2hk(d-1) + h^2 d(d-1)}{h^2 (h+1)^2}. \quad (45)$$

By plugging (43), (44), and (46) into (42), we obtain

$$\frac{(1-h) \{H\}_m (hd+m)}{1 - \{H\}_m}$$
$$\ge \frac{(k-k^2) - 2hk(d-1) + h^2((2d-1)k + 2d - d^2) + h^3 d^2}{hd+k}.$$

$$r(1) = \sum_\alpha \binom{d-m}{\alpha} \left(d(m+\alpha-1) - 2(1+2+\cdots+(m+\alpha-1)) \right)$$

$$= \sum_\alpha \binom{d-m}{\alpha} \left(d(m+\alpha-1) - (m+\alpha-1)(m+\alpha) \right)$$

$$= \sum_\alpha \binom{d-m}{\alpha} \left((-d-m+dm+m^2) + \alpha(d+2m-1) + \alpha^2 \right). \tag{50}$$

The derivative of the right-hand side with respect to k is equal to

$$\frac{h-1}{(hd+k)^2} \left(k^2 + hd(2k-1) + h^2 d(d-1) \right)$$

which is negative if $k > 1/2$. Therefore, the right-hand side attains its maximum for $k = 0$ or $k = 1$. The difference for $k = 0$ and $k = 1$ is

$$\frac{(d-1)(1-h)}{1+hd} h \ge 0$$

and, thus, the maximum is attained at $k = 0$ and we need to prove that

$$\frac{(1-h)\{H\}_m (hd+m)}{1 - \{H\}_m} \ge \frac{h^2(2d-d^2) + h^3 d^2}{hd}.$$

This is equivalent to

$$hd \left(2^{d-m} - (h+1)^{d-m} h^m \right)$$
$$\ge 2h \frac{2^{d-m} - (h+1)^{d-m} h^m}{1-h} - (h+1)^{d-m} h^m (hd+m). \tag{46}$$

Using the fact that

$$(h+1)^{d-m} h^m = \sum_{\alpha=0}^{d-m} \binom{d-m}{\alpha} h^{m+\alpha}$$

and (39), the left-hand side of (46) can be rewritten as

$$hd \left(\sum_\alpha \binom{d-m}{\alpha} - \sum_\alpha \binom{d-m}{\alpha} h^{m+\alpha} \right)$$
$$= hd \sum_\alpha \binom{d-m}{\alpha} (1 - h^{m+\alpha})$$
$$= d(1-h) \sum_\alpha \binom{d-m}{\alpha} (h + h^2 + \cdots + h^{m+\alpha}). \tag{47}$$

Similarly, the right-hand side of (46) is equal to

$$2h \sum_\alpha \binom{d-m}{\alpha} (1 + h + \cdots + h^{m+\alpha-1})$$
$$- \sum_\alpha \binom{d-m}{\alpha} h^{m+\alpha}(hd+m)$$
$$= \sum_\alpha \binom{d-m}{\alpha} (2(h+h^2+\cdots+h^{m+\alpha}) - h^{m+\alpha}(hd+m)).$$

Using that by (39) and (40)

$$\sum_\alpha \binom{d-m}{\alpha}(d-m-2\alpha) = 0 \tag{48}$$

after some mathematical manipulations, we obtain

$$\sum_\alpha \binom{d-m}{\alpha} \left(2(h+h^2+\cdots+h^{m+\alpha}) \right.$$
$$\left. - h^{m+\alpha}(hd+m) + (d-m-2\alpha) \right)$$
$$= (1-h) \sum_\alpha \binom{d-m}{\alpha}$$
$$\times \left(2h(1+2h+\cdots+(m+\alpha-1)h^{m+\alpha-2}) + h^{m+\alpha}d \right). \tag{49}$$

Finally, comparing expansions for the left-hand side and the right-hand side, that is (47) and (49), respectively, (46) becomes

$$\sum_\alpha \binom{d-m}{\alpha} d(1 + h + \cdots + h^{m+\alpha-2})$$
$$\ge 2 \sum_\alpha \binom{d-m}{\alpha} (1 + 2h + \cdots + (m+\alpha-1)h^{m+\alpha-2}).$$

Combining all the coefficients for each power of h, the left-hand side can be written as $r(h) = c_0 + c_1 h + \cdots + c_{d-2}h^{d-2}$ where

$$c_l = (d - 2(l+1)) \sum_{\alpha \ge (2+l-m) \vee 0}^{d-m} \binom{d-m}{\alpha}.$$

Notice that the sign of c_l is determined by $d - 2(l+1)$. Thus, we can define l_0 such that $c_l > 0$ if $l \le l_0$ and $c_l \le 0$ if $l > l_0$. Since $h \le 1$, this gives

$$r(h) = c_0 + c_1 h + \cdots + c_{d-2}h^{d-2}$$
$$= (c_0 + c_1 h + \cdots + c_{l_0}h^{l_0})$$
$$\quad + (c_{l_0+1}h^{l_0+1} + \cdots + c_{d-2}h^{d-2})$$
$$\ge (c_0 + c_1 + \cdots + c_{l_0})h^{l_0} + h^{l_0+1}(c_{l_0+1} + \cdots + c_{d-2})$$
$$\ge h^{l_0}(c_0 + c_1 + \cdots + c_{d-2}) = h^l r(1).$$

It remains to show that $r(1) \ge 0$. We observe that $r(1)$ results in (50) shown at the top of the page. Then, using (39)–(41), we obtain

$$r(1) = 2^{d-m-2}[4(-d-m+dm+m^2) + 2(d+2m-1)$$
$$\times (d-m) + (d-m-1)(d-m)]$$
$$= 3d^2 - 5d + 3(d-1)m + dm + m^2$$

which is positive for $d > 1$. $\qquad\square$

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