On the performance and the optimization of LDPC Codes on Fading Channels with Imperfect CSI

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***Abstract*—In this article we investigate low-density parity-check (LDPC) codes for fast Rayleigh fading channels in the presence of channel estimation error. The analysis is carried out using the Density Evolution technique assuming a belief propagation (BP) decoder. After having derived the analytical framework for decoding analysis, we prove that there exists a simple approximated relationship, function of estimation error variance, between the signal-to-noise ratio thresholds in the two cases of perfect and imperfect channel state information (CSI). This approximation follows the exact computation tightly, when the variance of the estimation error is sufficiently smaller than the fading variance. As a result, optimized codes for perfect CSI are also approximately optimized for imperfect CSI.**

I. Introduction

LDPC codes decoded with Belief Propagation (BP) algorithm achieve the optimum performance when the channel is completely known at the receiver [1], [2]. In real life however, channel needs to be estimated in some way and channel state estimation is never ideal. Thus, in analyzing the performance of BP decoder, channel estimation error must be kept into consideration. Since LDPC coded system are expected to operate at low signal-to-noise ratio, the estimation of envelope and phase of the channel is more critical and the quality of CSI plays a key role in the decoding performance.

In the literature the behavior of LDPC coded transmission in fading channels in the two opposite cases of perfect knowledge of CSI, or no knowledge of CSI has already been studied [3], [4]. The main tool used to carry out the asymptotic analysis of iterative decoding is the Density Evolution [1], which consists in tracking the evolution of probability density function (pdf) of the extrinsic information messages during the decoding and allows the computation of the decoding threshold. The intermediate case of partial knowledge of CSI is investigated in [5] for LDPC coded BICM by modeling CSI in the case of pilot symbol assisted modulation. With respect to the literature, in this article we model the imperfect CSI by introducing a complex channel estimation error, and apply Density Evolution by deriving closed form expressions for the pdf of initial decoding message.

In the rest of the paper we introduce the system model in Section II and derive analytical formulations of initial message pdf in Section III. In Section IV we introduce a closed form approximation of initial message pdf in the case of imperfect CSI which allows us to derive a simple relationship between decoding threshold and estimation error variance. Numerical results in Section V aims at verifying the accuracy of the proposed approximation.

II. System model

By assuming BPSK modulation, the received signal for a transmitted symbol $w = 1 - 2x$, $x \in \{0, 1\}$, will be:

$$y = \alpha w + n \qquad (1)$$

where n is the complex noise sample with variance $\sigma^2 = N_0/2$ per dimension and α is the complex fading coefficient. In this article we will consider only an uncorrelated flat fading channel, i.e. the channel obtained with fast fading variation along the codeword and ideal interleaving.

We will assume that channel state information (CSI) is available at the transmitter and that it is represented by $\hat{\alpha}$ statistically related to α. The case of no CSI will correspond to α and $\hat{\alpha}$ being mutually independent, whereas the case of perfect CSI will correspond to $\hat{\alpha} = \alpha$. The former two cases have already been investigated in literature, e.g. in [4]. In this paper we will consider the intermediate case of imperfect CSI where the fading coefficient is $\hat{\alpha} = \alpha + e$ where e obey to a complex Gaussian distribution with zero mean and variance σ_e^2 per dimension.

Up to date, the best known decoding algorithm for LDPC codes is Belief Propagation. BP decoding is performed by passing messages along the edges of the Tanner graph from variable nodes to their incident check nodes and back. Messages are typically real-valued but they can also assume the values $\pm\infty$, meaning that bits are known with absolute certainty. A message is a log-likelihood ratio of the form:

$$m = ln \frac{p(x = 1|y, \hat{\alpha})}{p(x = 0|y, \hat{\alpha})} = ln \frac{p(y|x = 1, \hat{\alpha})}{p(y|x = 0, \hat{\alpha})} \qquad (2)$$

where x is the random variable describing a codeword bit and y is the random variable describing all the information incorporated into this message (see [1]). By $m_{vc}^{(\ell)}$ we denote the message sent from variable node v to its incident check node c at the iteration step l, whereas by $m_{cv}^{(\ell)}$ we denote the message passed from check node c to its incident variable node v (see [1] and [2]).

To better describe BP decoding it is customary to define the map $\gamma : [-\infty, +\infty] \rightarrow GF(2) \times [0, +\infty]$ as follows. Given $z \in [-\infty, +\infty], z \neq 0$

$$\gamma(z) := (sgn\, z, -\ln\tanh|\frac{z}{2}|)\,. \quad (3)$$

Let m_v be the log-likelihood ratio of the codeword bit x associated to the variable node v based only on the channel observation of this bit. The update equations for the messages under BP are the following:

$$m_{vc}^{(\ell)} = \begin{cases} m_v, & \text{if } \ell = 0, \\ m_v \ + \sum_{c' \in C_v \setminus \{c\}} m_{c'v}^{(\ell-1)} & \text{if } \ell \geq 0 \end{cases} \quad (4)$$

$$m_{cv}^{(\ell)} = \gamma^{-1}(\sum_{v' \in V_c \setminus \{v\}} \gamma(m_{v'c}^{\ell-1})) \quad (5)$$

where C_v (V_c) is the set of check (variable) nodes connected to variable (check) node v (c).

In analyzing the performance of an LDPC code, it is usual not to focus on the performance of the single code, but rather to study the average performance of a code ensemble. All codes belonging to the same ensemble are constructed randomly according to the same two degree distributions $\lambda(x)$ and $\rho(x)$ [1]. In particular a degree distribution $\gamma(x)$ is a polynomial of maximum degree d of the form:

$$\lambda(x) = \sum_{i \geq 2}^{d} \lambda_i x^{i-1} \quad (6)$$

with non-negative coefficients and such that $\lambda(1) = 1$. A code ensemble is specified by the degree distribution pair (λ, ρ) where λ_i (ρ_i) represents the fraction of edges emanating from variable (check) nodes of degree i.

Once the number of iterations of an iterative (BP) decoder is fixed, the performance of a single LDPC code concentrates around the average performance of the ensemble. Since this concentration is exponential with the code-length, for sufficiently long codes the average performance is a realistic performance index. Since LDPC codes are linear codes, the performance does not depend on the transmitted codeword and, by convention, the analysis is performed assuming the transmission of the all-zero codeword.

Messages initially emitted by variable nodes are determined by the communication channel and have an associated error probability. By tracking how message distributions evolve under the iteration of BP decoder, we can discover the evolution of error probability as a function of the iteration number. This procedure of tracking the evolution of message densities is named *Density Evolution* (see [1] and [2]). By applying this method, given a degree distribution pair (λ, ρ), it is possible to determine the threshold signal-to-noise ratio above which the error probability fall to zero.

Good irregular LDPC optimized with respect to threshold have been found in [4] for uncorrelated Rayleigh fading channel with perfect CSI. Here we address the more general case of imperfect CSI by reformulating message distribution, by evaluating how code thresholds change and by finding optimized distribution pairs. The analytical framework which has to be set up for the evaluation of the density evolution requires the definition of the initial message m_0 at the decoder, with the evaluation of its pdf $f_{M_0}(m_0)$ and the verification of the stability conditions.

III. Evaluation of the initial message PDF

With the assumption of fast Rayleigh fading, α is an uncorrelated r.v along the codeword, with complex Gaussian distribution, i.e. its envelope $|\alpha| = R$ is Rayleigh distributed with pdf

$$f_R(r) = \frac{r}{\sigma_R{}^2}\exp(-\frac{r^2}{2\sigma_R{}^2}) \quad (7)$$

In case of perfect CSI we assume $\hat{\alpha} = \alpha$ and $|\hat{\alpha}| = R$; the initial message at the BP decoder is

$$m_0 = \ln\frac{\exp|y-\alpha|^2/2\sigma^2}{\exp|y+\alpha|^2/2\sigma^2} = \frac{2}{\sigma^2}\mathrm{Re}[y\alpha^*] \quad (8)$$

where $y = \alpha + n$ when assuming the transmission of the all-zero codeword. It is easy to see that m_0, when conditioned to $|\alpha| = R$, is a Gaussian r.v. with mean $2R^2/\sigma^2$ and variance $4R^2/\sigma^2$, i.e.

$$f_{M_0|R}(m_0|r) = \frac{\sigma}{2r\sqrt{2\pi}}\exp\left[-\frac{(m_0 - 2r^2/\sigma^2)^2}{8r^2/\sigma^2}\right] \quad (9)$$

The unconditional distribution of m_0 can be obtained in closed form as follows (the details are illustrated in the Appendix):

$$\begin{aligned} f_{M_0}(m_0) &= \int_0^{+\infty} f_{M_0|R}(m_0|r) f_R(r) dr & (10) \\ &= \frac{\sigma^2}{4\sigma_R\sqrt{\sigma^2+\sigma_R^2}} e^{\frac{1}{2}\left(m_0 - |m_0|\sqrt{1+\frac{\sigma^2}{\sigma_R^2}}\right)} & (11) \\ &= f_0(m_0; \sigma_R^2/\sigma^2) & (12) \end{aligned}$$

where $\sigma_R^2/\sigma^2 = R_c E_b/N_o$, being R_c the code rate. This closed form is novel with respect to the literature, e.g. [4], and emphasizes the direct dependence of message densities on signal-to-noise ratio.

As far as stability condition is concerned, as also shown in [4] the pdf in (12) verifies the symmetry property (see [1]) and the stability condition can be derived as $\lambda'(0)\rho'(1) < 1 + \sigma_R^2/\sigma^2$.

In the case of imperfect CSI the receiver will measure $\hat{\alpha}$ instead of α introducing therefore some error. If the decoder has a statistical knowledge of the error, i.e. it knows σ_e^2, it can operate with the initial messages of the BP decoder in the form:

$$m_0 = ln\frac{p(y,\hat{\alpha}|x=1)}{p(y,\hat{\alpha}|x=0)} = \frac{2}{\sigma^2(1+\sigma_e^2/\sigma_R^2)+\sigma_e^2}\mathrm{Re}[y\hat{\alpha}^*] \quad (13)$$

where the last expression is obtained after suitable manipulation of the ratio of two multivariate Gaussian distributions. However, if the decoder does not know the error and fading variances σ_e^2 and σ_R^2, it simply operates with $m_0 = (2/\sigma^2)\mathrm{Re}[y\hat{\alpha}^*]$. Now, by taking into account that, given α and e, the r.v. m_0 is Gaussian with mean $K(R^2 + \mathrm{Re}[\alpha e*])$ and variance $K^2\sigma^2|\alpha+e|^2$ (here, $K = 2/[\sigma^2(1+\sigma_e^2/\sigma_R^2)+\sigma_e^2]$):

$$f_{M_0|\alpha,e}(m_0|\alpha,e) = \frac{1}{K\sigma|\alpha+e|\sqrt{2\pi}} \exp\left[-\frac{(m_0 - K(R^2+\mathrm{Re}[\alpha e*]))^2}{2K^2\sigma^2|\alpha+e|^2}\right] \quad (14)$$

The unconditioned density $f_{M_0}(m_0)$ can be obtained by full integration over the domain of the two complex r.v. α and e. However, by introducing the angle θ between the two complex variables α and e, we can replace $|\alpha+e|^2$ with $R^2 + E^2 + 2RE\cos(\theta)$ and $\mathrm{Re}[\alpha e*]$ with $RE\cos(\theta)$, thus deriving the unconditional pdf as:

$$f_{M_0}(m_0) = \iiint dr\, dE\, d\theta\, f_{M_0|\alpha,e}(m_0|\alpha,e)\,\frac{1}{2\pi}\, f_E(E)\, f_R(r) \quad (15)$$

where $f_E(E)$ is the Rayleigh pdf of the envelope E of r.v. e. By looking carefully inside the last equation, we can observe that pdf is a function of the two parameters $\sigma_R^2/\sigma^2 = R_c E_b/N_o$ and $\sigma_e^2/\sigma_R^2 = \eta$, where the latter is the normalized error variance.

Expressions in (12) and (15) can be used in the Density Evolution procedure to compute performance thresholds with both perfect and imperfect CSI.

IV. Approximated evaluation with imperfect CSI

In this section we explore the possibility of deriving a closed form approximation of the initial message pdf which avoids the triple integral in (15). Looking at m_0 in (13), we can write it as

$$m_0 = K\{R^2 + \mathrm{Re}[e\alpha^*] + \mathrm{Re}[\alpha n^*] + \mathrm{Re}[en^*]\} \quad (16)$$

When $\sigma_e^2 << \sigma_R^2$, i.e. $\eta << 1$, the initial message m_0 can be approximated by considering $\mathrm{Re}[en^*] \approx 0$ and $K \approx 1/(\sigma^2 + \sigma_e^2)$. In this case, the r.v. m_0 conditioned to α becomes a Gaussian r.v. with mean $2(\sigma^2+\sigma_e^2)^{-1}R^2$ and variance $4(\sigma^2 + \sigma_e^2)^{-1}R^2$. Its pdf becomes:

$$f_{M_0|R}(m_0|r) = \frac{\sqrt{\sigma^2+\sigma_e^2}}{2r\sqrt{2\pi}} \exp\left[-\frac{(m_0 - 2r^2/(\sigma^2+\sigma_e^2))^2}{8r^2/(\sigma^2+\sigma_e^2)}\right] \quad (17)$$

which can be easily integrated, as in (12), to obtain:

$$f_{M_0}(m_0) = \frac{\frac{\sigma^2+\sigma_e^2}{\sigma_e^2+\sigma_R^2}\, e^{\frac{1}{2}\left(m_0 - |m_0|\sqrt{1+\frac{\sigma^2+\sigma_e^2}{\sigma_e^2+\sigma_R^2}}\right)}}{4\sqrt{1+\frac{\sigma^2+\sigma_e^2}{\sigma_e^2+\sigma_R^2}}} \quad (18)$$

$$= f_0\left(m_0; \frac{1}{\sigma^2/\sigma_R^2 + \eta}\right) \quad (19)$$

Since this expression shows the same functional dependence of eq. (12), i.e. $f_0(.)$, it also tells us that if a code has a threshold T with perfect CSI, it has in presence of imperfect CSI a threshold T_I satisfying the equation:

$$R_c T = 1/(1/R_c T_I + \eta) \quad (20)$$

meaning that

$$T_I = 1/(1/T_I - R_c\eta) \approx T(1 + \eta R_c T) \quad (21)$$

The threshold increases approximately linearly with η and its slope is large when T is large, i.e. powerful codes are also more robust to CSI estimation error. Moreover, since the relationship between T_I and T is monotonic, an optimized code within a given class of codes with perfect CSI, e.g. the optimized irregular codes in [4], is also quite optimized for a decoder with imperfect CSI if $\sigma_e^2 << \sigma_R^2$.

As far as stability condition is concerned, the pdf in (19) verifies the symmetry property (see [1]), i.e. $f_{M_0}(-m_0)e^{m_0} = f_{M_0}(m_0)$, and the stability condition can be easily derived (because eq.(12) has the same form of eq.(19)) as

$$\lambda'(0)\rho'(1) < \frac{1}{\sigma^2/\sigma_R^2 + \eta}$$

This condition is verified for all the optimized irregular codes proposed in [4].

V. Numerical results

The numerical results provided in this section aim at checking and discussing the proposed approximation for closed form evaluation of decoding threshold with imperfect CSI.

In Fig.1 we first present the approximated pdf of the initial decoding message and compare it with the exact one. The figure is obtained using $\eta = 0.5$ just to make the differences visible. We can see that the left tail is reduced while the the values around zero are increased.

In Fig.2 we compare the decoding thresholds obtained with Density Evolution in one case and with the approximated formula in (21), as a function of η The figure shows how the threshold T_I increases as a function of η for the rate 1/2 optimized irregular code with maximal left degree equal to 10, which achieves a threshold T(dB)=2.08 [4] in case of perfect CSI. We note that the approximated evaluation follows quite closely the exact evaluation for the smallest values of η.

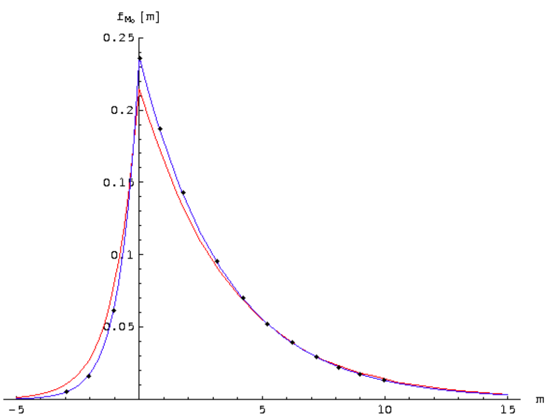

Fig. 1. The pdf of m_0 drawn with $\sigma_R^2/\sigma^2 = 1$ and $\eta = 0.5$ compared with the approximation (curve with diamonds) as in (19).

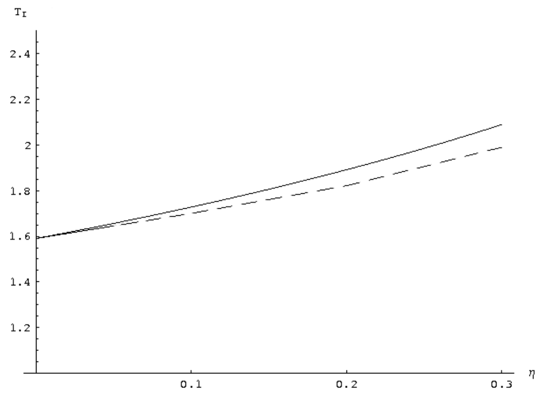

Fig. 2. Threshold of optimized rate-1/2 irregular code as in [4] (with maximal left degree of 10), as a function of η. The solid line is obtained with the approximation as in (21).

VI. Conclusions

In this paper we have investigated the decoding threshold of LDPC codes in fast Rayleigh fading channel with imperfect CSI. We have derived an analytical framework for the evaluation of both exact and approximated expressions of the initial message distribution. Starting from the approximated expression we have derived a simple relationship for the evaluation of the decoding threshold with imperfect CSI given the threshold for perfect CSI and the value of the normalized error variance η. The results obtained follow quite closely the exact computation. Moreover, due to the structure of this relationship, we have inferred that optimized codes for perfect CSI are also approximately optimized for imperfect CSI.

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Appendix

Here, we derive the closed form es pression reported in (12) of the text. We start with the expression of $f_{M_0}(m_0)$ by substituting the conditional pdf of m_0 and the pdf of R:

$$\begin{aligned} f_{M_0}(m_0) &= \int_0^{+\infty} f_{M_0|R}(m_0|r) f_R(r)dr \\ &= \int_0^{+\infty} \frac{\sigma}{2r\sqrt{2\pi}} e^{\frac{-\left(m_0-2r^2/\sigma^2\right)^2}{8r^2/\sigma^2}} \cdot \frac{r}{\sigma_R^2} e^{-\frac{r^2}{2\sigma_R^2}} dr \\ &= \int_0^{+\infty} \frac{\sigma}{2\sqrt{2\pi}\sigma_R^2} e^{-\frac{1}{8}\left(\frac{4}{\sigma_R^2}+\frac{4}{\sigma^2}\right)r^2+\frac{m}{2}-\frac{m^2\sigma^2}{8r^2}} dr \end{aligned} \tag{22}$$

By introducing and the following parameters: $A = \frac{m^2\sigma^2}{8}$, $B = \frac{1}{2}\left(\frac{1}{\sigma^2}+\frac{1}{\sigma_R^2}\right)$, $C = \frac{\sigma e^{\frac{m}{2}}}{2\sqrt{2\pi}\sigma_R^2}$, we obtain

$$f_{M_0}(m_0) = C\int_0^{+\infty} e^{-\frac{A}{r^2}-Br^2} dr \tag{23}$$

$$\begin{aligned} &= Ce^{2\sqrt{AB}} \int_0^{+\infty} e^{-\left(\frac{\sqrt{A}}{r}+\sqrt{B}r\right)^2} dr \\ &= \frac{Ce^{2\sqrt{AB}}}{\sqrt{B}} \int_0^{+\infty} \frac{\sqrt{B}}{2} e^{-\left(\frac{\sqrt{A}}{r}+\sqrt{B}r\right)^2} + \\ &+ \frac{\sqrt{B}}{2} e^{-\left(\frac{\sqrt{A}}{r}+\sqrt{B}r\right)^2} + \frac{\sqrt{A}}{2r^2} e^{-\left(\frac{\sqrt{A}}{r}+\sqrt{B}r\right)^2} + \\ &- \frac{\sqrt{A}}{2r^2} e^{-\left(\frac{\sqrt{A}}{r}+\sqrt{B}r\right)^2} dr \\ &= \frac{Ce^{2\sqrt{AB}}}{\sqrt{B}} \int_0^{+\infty} e^{-\left(\frac{\sqrt{A}}{r}+\sqrt{B}r\right)^2} \left(\sqrt{B}-\frac{\sqrt{A}}{r^2}\right) + \\ &+ e^{-4\sqrt{AB}} e^{-\left(-\frac{\sqrt{A}}{r}+\sqrt{B}r\right)^2} \left(\sqrt{B}+\frac{\sqrt{A}}{r^2}\right) dr \\ &= \frac{\sqrt{\pi}}{2} \frac{Ce^{2\sqrt{AB}}}{\sqrt{B}} Erf\left[\frac{\sqrt{A}}{r}+\sqrt{B}r\right]_0^{+\infty} + \\ &+ \frac{\sqrt{\pi}}{2} \frac{Ce^{2\sqrt{AB}}}{\sqrt{B}} e^{-4\sqrt{AB}} Erf\left[-\frac{\sqrt{A}}{r}+\sqrt{B}r\right]_0^{+\infty} \end{aligned}$$

$$= \frac{\sigma e^{\frac{m}{2}}}{2\sqrt{2}\sigma_R^2} \frac{e^{-2\sqrt{AB}}}{2\sqrt{B}} \tag{24}$$

$$= \frac{\frac{\sigma^2}{\sigma_R^2}}{4\sqrt{1+\frac{\sigma^2}{\sigma_R^2}}} e^{\frac{1}{2}\left(m_0-|m_0|\sqrt{1+\frac{\sigma^2}{\sigma_R^2}}\right)} \tag{25}$$

Moreover:

$$\int_{-\infty}^{0} \frac{\frac{\sigma^2}{\sigma_R^2}}{4\sqrt{1+\frac{\sigma^2}{\sigma_R^2}}} e^{\frac{1}{2}\left(m_0-|m_0|\sqrt{1+\frac{\sigma^2}{\sigma_R^2}}\right)} dm_0 = \frac{1}{2} - \frac{1}{2\sqrt{1+\frac{\sigma^2}{\sigma_R^2}}}$$